Journal of Engineering Mathematics, Vol. 6, No. 4, October 1972
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

A Perturbation Method for the Radiation of Surface Waves

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(Received January 3, 1972)

SUMMARY
In this paper we derive a straight forward asymptotic method to find the wave solution for the case that a circular cylinder is heaving in a free surface. The wave period is supposed to be small. The methods used are similar to methods used in the theory of geometrical optics and the theory of boundary layer expansions. It turns out that not only the lowest order approximation can be easily calculated, higher order approximations follow as well.

1. Introduction

This work concerns a problem which has partly been solved by Ursell [7] in 1953. Ursell considers a long circular cylinder with its axis horizontal. This cylinder is half-immersed in a fluid under gravity and is making periodic vertical oscillations of small constant amplitude about this position. Ursell tackled this problem in a straight forward way. He derives an integral equation for the potential function on the cylinder. This integral equation is of a rather complicated form. In order to get some insight in the solution of this equation, it is obvious that, because this solution cannot be obtained explicitly, an asymptotic expansion with respect to the short wave length is the only way out. Ursell obtains such an asymptotic expansion by introducing a Green's function which leads to a small kernel. The choice of this Green's function is not as obvious as may be expected. Because the method is very complicated, we may pose the question whether there are no other methods to solve this problem. But there are other reasons to look for a different method. In his concluding remarks Ursell states that it has not yet been shown how the method can be extended to general three-dimensional problems. Up to now there is no answer to this important question and we are strengthened in our opinion that there is a need for an asymptotic method without these limitations. Such a method will be explained in this paper, although we treat a simple two-dimensional problem. It will be clear that by using the same reasoning a general three-dimensional problem can be treated. An advantage of treating the same problem as Ursell did is that the results can be compared easily. We have to do it this way because no general proof of validity will be given for our method. It is generally known that there is a need for such a comparison in an asymptotic approach of this kind of problems, see Keller [5].

As we mentioned before the method we employ in this paper is completely different compared with Ursell's approach. We introduce the concept of inner and outer expansion for this problem. We do this similar to expansions given by Van Dyke [2] and Cole [1]. However, because of the geometric interpretation of inner and outer expansion not being clear at first, we use the terminology local and regular approximation. This will be clear later on. Ursell [8] remarks that it is not possible to make straight forward expansions of the potential in inverse powers of the wave number k, because of the exponential behavior of the wave train $\exp\{k(y+ix)$. It is obvious that it is possible to make expansions similar to geometrical optics (Keller [5]).

Hence we will obtain expansions of the form

$$\sum_n A_n(x, y) \exp\{k(y+ix)\}$$

where

$$A_n = o(A_{n-1}) \quad \text{for} \quad k \to \infty \ .$$

2. Formulation of the Problem

We assume the depth of the water to be infinite although the methods can be applied for the finite water depth problem. The viscosity is negligible and the wave amplitude is small, i.e. the heave amplitude of the cylinder has to be small. It follows immediately that linearized equations are applicable to this problem. Attention will be confined to two-dimensional problems where a single obstacle intersects the free surface and is forced to heave motions only. For simplicity's sake it will be assumed that the obstacle is an infinitely long circular cylinder with horizontal generators. The cylinder is forced to perform a prescribed oscillatory vertical motion. The amplitude of this heave motion is supposed to be small and the frequency is large. Outgoing waves will be produced and the goal of this paper is to determine an approximation for these waves (See Fig. 1).

Let the radius of the circle be denoted by a. The period of the heave motion i.e. of the outgoing waves is denoted by $2\pi/\omega$ and we write $k=\omega^2/g$, $N=\omega^2 a/g=ka$ where g is the gravitational acceleration and a is of order unity.

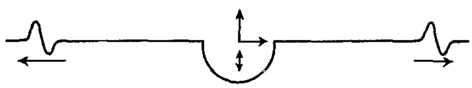

Fig. 1

The irrotational motion of an ideal fluid can be expressed in terms of a velocity potential $\Phi_{(t)}$ which for two-dimensional time periodic waves has the representation

$$\Phi_{(t)}(x, y, t)=\text{Re}\,\{\Phi(x, y)e^{-i\omega t}\}\,. \tag{2.1}$$

It is assumed that this potential exists and that the free surface condition may be linearized. We notice that this is a linearization with respect to the small amplitude. An other simplification occurs, if we neglect higher harmonics in the potential which are induced by the heave motion with frequency ω. The higher harmonics are not discussed in this paper.

The potential Φ satisfies:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0 \qquad \text{in the fluid} \tag{2.2}$$

$$\frac{\partial\Phi}{\partial y}-k\Phi=0 \qquad \text{at } y=0,\ |x|>a \tag{2.3}$$

$$\frac{\partial\Phi}{\partial r}=-U\sin\theta \qquad \text{at } r=a \text{ and } \pi<\theta<2\pi \tag{2.4}$$

where r, θ are polar coordinated defined by

$$x=r\cos\theta,\quad y=r\sin\theta.$$

The radiation condition tells us that the waves are outgoing.

We will look for an approximation of the potential function if $k\gg 1$. In this case the wave contribution is asymptotically non-zero inside a thin layer near the free surface (layer thickness is $O(1/k)$ for $k\to\infty$). Therefore if we expand the velocity potential in inverse powers of k, we know that this approximation does not give a wave contribution. This can be easily seen because the free surface condition reduces to $\Phi=0$ on the free surface. However it leads to the correct vertical fluid velocity near the free surface and the circular cylinder. Hence it serves as a regular solution for the wave problem and the waves follow from the local solution which will be found for small values of y.

We will now describe the method which gives us the regular solution. We suppose $\Phi(x, y, k)$ is a regular asymptotic power series in k^{-1}

$$\Phi(x, y, k) = \psi_0(x, y) + \frac{1}{k}\psi_1(x, y) + \dots \tag{2.5}$$

The functions ψ_i are functions of the coordinates only. We will give the general method for the construction of ψ_i although it is clear that for the circular case solutions can be found by simpler techniques.

The equation for $\psi_i(x, y)$ becomes:

$$\frac{\partial^2\psi_i}{\partial x^2} + \frac{\partial^2\psi_i}{\partial y^2} = 0 \qquad \text{in the fluid} \tag{2.6}$$

$$\psi_i = \frac{\partial\psi_{i-1}}{\partial y} \qquad \text{at } y=0,\ |x|>a \tag{2.7}$$

$$\frac{\partial\psi_i}{\partial r} = -\delta_i^0 U \sin\theta \qquad \text{at } r=a \text{ and } \pi<\theta<2\pi \tag{2.8}$$

where $\delta_i^0 = \begin{cases} 1 & \text{if } i=0 \\ 0 & \text{if } i\neq 0 \end{cases}$ and $\psi_i \equiv 0$ if $i<0$.

This problem can be considered as a singular perturbation problem because (2.7) is a lower order condition than (2.3). For this reason a boundary layer may be expected near $y=0$, where we cannot fulfil all conditions. In turns out that we cannot obey the wave condition. Therefore we take as a condition at infinity

$$\lim_{|x|\to\infty \text{ or } |y|\to\infty} \psi_i(x, y) = 0 . \tag{2.9}$$

To find the solution of (2.6) with conditions (2.7)–(2.9), we construct a Green's function $G(x, y, \xi, \eta)$ which is a solution of

$$\Delta G = 2\pi\,\delta(x-\xi)\,\delta(y-\eta) \qquad -\infty<x<\infty,\ -\infty<y<0$$

with $G=0$ at $y=0$ and $G\to 0$ if $|x|\to\infty$ or $|y|\to\infty$.

It follows that this Green's function equals

$$G(x, y, \xi, \eta) = \tfrac{1}{2}\ln\left[\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}\right] \tag{2.10}$$

and the solution of (2.6)–(2.9) can be written as

$$\begin{aligned} 2\pi\psi_i(x, y) = a\int_\pi^{2\pi} \Big\langle \Big\{ &\frac{\partial\psi_i}{\partial\rho}(\rho\cos\alpha, \rho\sin\alpha)\Big\rangle_{\rho=a} \times \\ &\times G(x, y; a\cos\alpha; a\sin\alpha) - \psi_i(a\cos\alpha, a\sin\alpha)\times \\ &\times\Big\langle\frac{\partial G}{\partial\rho}(x, y; \rho\cos\alpha, \rho\sin\alpha)\Big\rangle_{\rho=a}\Big\} d\alpha + \int_{-\infty}^{-a}\psi_i(\xi, 0)\Big\langle\frac{\partial G}{\partial\eta}(x, y; \xi, \eta)\Big\rangle_{\eta=0} d\xi + \\ &+\int_a^\infty \psi_i(\xi, 0)\Big\langle\frac{\partial G}{\partial\eta}(x, y; \xi, \eta)\Big\rangle_{\eta=0} d\xi . \end{aligned} \tag{2.11}$$

This equation is still an integral equation because in the integral over the circle the function ψ_i is still unknown.

However this equation can easily be solved because $\langle(\partial G/\partial\rho)(x, y, \rho\cos\alpha, \rho\sin\alpha)\rangle_{\rho=a}=0$ for $x=a\cos\theta$, $y=a\sin\theta$. Hence if we consider a more general shape, we have to take an other Green's function with this property on the surface. So in the case of a circular cylinder we get

$$\pi\psi_i(a\cos\theta,\ a\sin\theta)=a\int_{\pi}^{2\pi}\left\langle\frac{\partial\psi_i}{\partial\rho}(\rho\cos\alpha,\ \rho\sin\alpha)\right\rangle_{\rho=a}G(\theta,\alpha)d\alpha$$
$$+\int_{-\infty}^{-a}\psi_i(\xi,0)\left\langle\frac{\partial G}{\partial\eta}(\theta;\xi,\eta)\right\rangle_{\eta=0}d\xi+\int_{a}^{\infty}\psi_i(\xi,0)\left\langle\frac{\partial G}{\partial\eta}(\theta;\xi,\eta)\right\rangle_{\eta=0}d\xi\,,\quad(2.12)$$

where we used an obvious notation in the arguments of the Green's function.

We must pay some special attention to the properties of $\psi_0(x, y)$ near the cylinder. From (2.11) follows that

$$\pi\frac{\partial\psi_0}{\partial y}=a\int_{\pi}^{2\pi}\left\{\left\langle\frac{\partial\psi_0}{\partial\rho}\right\rangle_{\rho=a}\frac{\partial G}{\partial y}-\psi_0\left\langle\frac{\partial^2 G}{\partial y\partial\rho}\right\rangle_{\rho=a}\right\}d\alpha \quad(2.13)$$

for x, y on the circle cylinder.

After some calculations it appears that for $x\sim a$ and $y\sim 0$

$$\frac{\partial\psi_0}{\partial y}\sim U$$

This can be derived by a conformal mapping which leads to the results much quicker, however the method of conformal mapping is more complicated for more general objects.

The velocity near the circle and the free surface is of order one and subsequent approximations can be calculated by (2.11). Like we expected no wave contribution can be found because the Green's function tends to zero at infinity.

3. The Wave Solution

In the preceding sections we mentioned that we may consider the boundary value problem as a singular perturbation problem, because the order of the free surface condition considered in section 2 is lower than the real one. Therefore we must stretch the y-coordinate to fulfil the complete free surface condition. On the other hand we like to find a solution for the potential equation. Hence we must stretch the x-coordinate as well.

The first requirement leads to the transformation

$$y'=ky\,. \quad(3.1)$$

Because we wish to determine the radiated wave and we want to take care of the boundary condition on the cylinder, we stretch the x-coordinate as follows

$$x'=k(x-a)\,. \quad(3.2)$$

This leads to the solution for $x>a$ and because of the symmetry with respect to $x=0$ the solution for $x<-a$ is known as well. In these new coordinates the equation for the circle becomes

$$x'\sim\frac{-y'^2}{2ka}+O\left(\frac{1}{(ka)^3}\right). \quad(3.3)$$

The equation and boundary conditions for $\Phi(x', y')$ become

$$\frac{\partial^2\Phi}{\partial x'^2}+\frac{\partial^2\Phi}{\partial y'^2}=0\quad\text{for}\quad\frac{-y'^2}{2ka}<x'<\infty\text{ and }\quad-\infty<y'<0 \quad(3.4)$$

$$\frac{\partial\Phi}{\partial y'}-\Phi=0\quad\text{at}\quad y'=0 \quad(3.5)$$

$$\frac{\partial\Phi}{\partial x'}+\frac{\partial\Phi}{\partial y'}\left\{\frac{y'}{ka}+O\left(\frac{1}{(ka)^3}\right)\right\}\sim\frac{-aUy'}{(ka)^2}\quad\text{at}\quad x'=\frac{-y'^2}{2ka} \quad(3.6)$$

radiation condition for $x'\to\infty$. (3.7)

From (3.6) follows that for large values of $N = ka$ the local potential function Φ_l can be written as an asymptotic series

$$\Phi_l(x', y') = \Phi_{0l}(x', y') + \Phi_{1l}(x', y') + \dots \tag{3.8}$$

where $\Phi_{il} = o(\Phi_{i-1,l})$ for N large

If we insert this in relation (3.6) and equate the lowest order term to zero we see that

$$\Phi_{0l}(x', y') = O\left\{\frac{1}{(ka)^2}\right\} \quad \text{and} \quad \frac{\partial \Phi_{0l}}{\partial y'} \text{ is of the same order.}$$

However from the calculations in Section 2 it follows that the water particles have a velocity $+U$ with respect to the fixed cylinder near the surface and the cylinder and because the stretching of the coordinates is a stretching with respect to the fixed circle cylinder, we have to deal with this relative velocity in (3.6). We know that

$$\frac{\partial \psi_0}{\partial y'} \sim + \frac{aU}{N},$$

therefore ψ_0 is the lowest order approximation in the whole fluid domain and must be added to (3.8), see O'Malley [3]. This leads to a contribution in the lowest order approximation of the wave potential. Hence condition (3.6) becomes up to the lowest order.

$$\frac{\partial \Phi_{0l}}{\partial x'} = \frac{-2aU}{N^2} y' \text{ at } x' = 0 \tag{3.9}$$

We now proceed with the determination of Φ_{0l}, which is a solution of

$$\frac{\partial^2 \Phi_{0l}}{\partial x'^2} + \frac{\partial^2 \Phi_{0l}}{\partial y'^2} = 0 \quad \text{for} \quad x' > 0,\ y' < 0 \tag{3.10}$$

$$\frac{\partial \Phi_{0l}}{\partial y'} - \Phi_{0l} = 0 \qquad \text{for} \quad y' = 0 \tag{3.11}$$

$$\frac{\partial \Phi_{0l}}{\partial x'} = \frac{-2aU}{N^2} y' \qquad \text{for} \quad x' = 0 \text{ and } y' \text{ finite} \tag{3.12}$$

It is worthwhile to pay more attention to condition (3.12). This condition does not meet the requirement that the velocity tends to zero if $y \to -\infty$. However for large values of y the solution of section 2 holds. Therefore (3.12) can only be used for the determination of the wave contribution which holds near the free surface. To find the wave solution of (3.10)–(3.12) we construct a Green's function which satisfies

$$g_{x'x'} + g_{y'y'} = \delta(x' - \xi)\,\delta(y' - \eta) \tag{3.13}$$

$$g_{y'} - g = 0 \quad \text{at} \quad y' = 0 \tag{3.14}$$

$$g_{x'} = 0 \qquad \text{at} \quad x' = 0 \tag{3.15}$$

radiation condition. (3.16)

First we disregard condition (3.15). This condition will be met by reflexion. We consider $g^0(x', y', \xi, \eta)$ which is a solution of (3.13), (3.14) and (3.16). This solution $g^0(x', y', \xi, \eta)$ is well known—see: John [4]—and may be written in the form:

$$g^0(x', y', \xi, \eta) = -i\,e^{i|x'-\xi| + (y'+\eta)} + \frac{1}{4\pi} \ln\left[\frac{(x'-\xi)^2 + (y'-\eta)^2}{(x'-\xi)^2 + (y'+\eta)^2}\right] + \tag{3.17}$$

$$-\frac{1}{\pi}\int_0^\infty \frac{t \cos(y'+\eta)t + \sin(y'+\eta)t}{1+t^2}\, e^{-|x'-\xi|t}\, dt$$

It can easily be shown that for large values of x' or ξ, g^0 behaves like

$$g^0(x', y', \xi, \eta) \sim -i\,e^{i|x'-\xi|+(y'+\eta)} + O\left|\frac{1}{x'-\xi}\right|.$$

From this function g^0 we can find g as follows

$$g(x', y', \xi, \eta) = g^0(x', y', \xi, \eta) + g^0(-x', y', \xi, \eta). \tag{3.18}$$

As we see (3.18) satisfies condition (3.15) and the solution is of the form

$$\Phi_{0l} = \frac{-4aU}{N^2}\int_{-\infty}^{0} \eta g^0(x', y', 0, \eta)\,d\eta \tag{3.19}$$

because $g^0(-x', y', \xi, \eta) = g^0(x', y', -\xi, \eta)$.

It is obvious that only for the wave part of Φ_{0l} the integration of η from $-\infty$ to 0 has a meaning. In other words the finite part of the integral has to be taken into account (see Keller [5]).

This leads to a wave part

$$\Phi_{0l} = \frac{4aUi}{N^2}\,e^{ix'+y'}\int_{-\infty}^{0} \eta\, e^{\eta}\,d\eta$$

$$= \frac{-4aUi}{N^2}\,e^{ix'+y'}. \tag{3.20}$$

For large values of x' the wave contribution (3.20) remains unaltered. Therefore, in the original coordinates we get

$$\Phi(x, y) \sim \frac{-4aUi}{N^2}\,e^{ik(x-a)+ky} \quad \text{for} \quad x \gg 0. \tag{3.21}$$

Because of the symmetry of the problem we get

$$\Phi(x, y) \sim \frac{-4aUi}{N^2}\,e^{ik(|x|-a)+ky} \quad \text{for large values of } |x|. \tag{3.22}$$

The surface amplitude at infinity is

$$\frac{i\omega\Phi(x, 0)}{g} = \frac{4\omega aU}{gN^2}$$

while the amplitude of motion of the cylinder equals $U\omega^{-1}$, and thus their ratio:

$$\text{wave making coefficient} = W_c = \frac{\text{wave amplitude at infinity}}{\text{amplitude of motion of the cylinder}} = \frac{4\omega^2 a}{gN^2} = \frac{4}{N}$$

This result agrees with the result derived by Ursell [7] and Rhodes–Robinson [6].

The paper of Rhodes–Robinson is an extension of [7] for the case of finite depth. In our theory this extension can be easily made by considering the depth of the water in the determination of the regular solution. We will not do so in the present paper, because it is more important to construct a higher order wave approximation as is clearly indicated in fig. 1 of [6].

4. Higher Order Approximation

In principle the theory of Ursell leads to higher order approximations. However, the derivation is even more lengthy than the derivation of (3.22) and neither carried out in [7] nor in [6]. With our method it is rather simple to make this extension. As follows from the preceding sections it is sufficient to find an approximation of $\partial\Phi/\partial x'$ in the vicinity of the cylinder.

After a thorough investigation of condition (3.6) it turns out that for the next order only

$\partial\psi_1/\partial y'$ is needed. Higher order approximations need other derivatives of ψ_0 also, since (3.6) is given on $x' = -y'^2/2ka$, and a complicated matching principle has to be used.

As we will see the asymptotic series we find is not a power series in N^{-1}. It is a well-known fact that logarithmic terms play an important role. The regular solution remains a series in powers of k^{-1}. In the preceding sections we suggested that $\psi_0(x, y)$ can be calculated by means of conformal mapping. If we do so, we find

$$\psi_0(x, y) = \frac{Ua^2 y}{x^2+y^2}. \tag{4.1}$$

In order to calculate $\psi_1(x, y)$ near the circle and the free surface, we use (2.12) together with (2.7). This yields

$$\pi\langle\psi_1(a\cos\theta, a\sin\theta)\rangle_{\theta\simeq 2\pi} \simeq -2ya^2U \times$$
$$\times \int_a^\infty \frac{1}{\xi^2}\left\{\frac{1}{(a-\xi)^2+y^2} + \frac{1}{(a+\xi)^2}\right\}d\xi = -2yU\left\{\frac{2}{a}\ln\frac{|y|}{2a} + \frac{5}{2a} + \frac{\pi/2}{y}\right\} \tag{4.2}$$

The derivative of ψ_1 with respect to y follows from (4.2) or by direct calculation.

We find

$$\left.\frac{\partial\psi_1}{\partial y}\right|_{x\sim a,\, y\sim a} \sim \frac{-U}{a}\left\{4\ln\frac{|y|}{a} - 4\ln 2+9\right\}. \tag{4.3}$$

For the wave contribution we must solve the problem

$$\frac{\partial^2\Phi_{1l}}{\partial x'^2} + \frac{\partial^2\Phi_{1l}}{\partial y'^2} = 0 \quad \text{for} \quad x'>0,\ y'<0 \tag{4.4}$$

$$\frac{\partial\Phi_{1l}}{\partial y'} - \Phi_{1l} = 0 \qquad \text{for } y'=0 \tag{4.5}$$

$$\frac{\partial\Phi_{1l}}{\partial x'} = +\frac{y'}{(ka)^3}\,Ua\{4\ln|y'| - 4\ln 2+9-4\ln ka\} \quad \text{for} \quad x'=0. \tag{4.6}$$

From condition (4.6) it follows that the second order wave approximation not only consists of a multiple of N^{-3} but of $N^{-3}\ln N$ as well. With the help of the Green's function (3.18) we find a wave contribution of the form

$$\Phi_{1l} = \tilde{\Phi}_{1l} + \tilde{\tilde{\Phi}}_1\ ,$$

where we split up Φ_{1l} into a $N^{-3}\ln N$ term and a N^{-3} term

$$\tilde{\Phi}_{1l} = -8aUi\,e^{ix'+y'}\frac{\ln N}{N^3} \tag{4.7}$$

and

$$\tilde{\tilde{\Phi}}_1 = \{26-8(\ln 2+\gamma)\}aUi\,\frac{e^{ix'+y'}}{N^3} \tag{4.8}$$

where $\gamma = 0.577$ is Euler's constant.

The second order approximation of the wave making coefficient becomes

$$W_c \simeq \frac{4}{N} + \frac{(8\ln N - 15.6)}{N^2}. \tag{4.9}$$

It is regrettable that no comparison with test results can be made because in the range of short waves no oscillation tests can be carried out. Vugts [9] shows that test results are reliable up to $N=1$. For frequencies with N between 1 and 2.25 an increase in the spread of results can be noticed.

The calculations carried out by Vugts are not valid in the short wave region.

5. Conclusion

The main result of this paper is that with a straight forward application of a perturbation technique results can be obtained for the radiation of short waves from an oscillating cylinder. Although the method is applied to a rather simple problem, we can make an extension to cylinders of general shape and water of finite depth. The calculations of Section 2 are a little more complicated, however, no principle difficulties occur. The only restriction we come across is the tangent to the obstacle near the free surface being perpendicular to the undisturbed free surface. We are not restricted to a parabolic approximation of the object in the stretched coordinates.

Acknowledgement

The author wishes to thank Prof. J. B. Keller of Courant Institute of Mathematical Sciences of New York University for the suggestion to apply straight forward expansion methods to water wave problems, while the author was visiting Courant Institute.

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